

GLUING LOCALLY SYMMETRIC MANIFOLDS: ASPHERICITY AND RIGIDITY

T. TÂM NGUYỄN PHAN

ABSTRACT. We use the reflection group trick to glue manifolds with corners that are Borel-Serre compactifications of locally symmetric spaces of noncompact type and obtain aspherical manifolds. We call these *piecewise locally symmetric* manifolds. This class of spaces provide new examples of aspherical manifolds whose fundamental groups have the structure of a complex of groups. These manifolds typically do not admit a locally CAT(0) metric. We prove that any self homotopy equivalence of such manifolds is homotopic to a homeomorphism. We compute the group of self homotopy equivalences of such a manifold and show that it can contain a normal free abelian subgroup, and thus can be infinite.

1. INTRODUCTION

There has been a desire to construct more examples of aspherical manifolds, to enrich the list of known examples or to disprove various conjectures on aspherical manifolds. Gluing constructions have been playing a crucial role in giving interesting examples of such manifolds. For example, the Davis construction using the reflection group trick gives examples of closed aspherical manifolds whose universal covers are not homeomorphic to any Euclidean spaces.

Piecewise locally symmetric manifolds. In this paper we introduce a class of aspherical manifolds, which we call *piecewise locally symmetric manifolds*, that are obtained by applying the reflection group trick to manifolds with corners that are the Borel-Serre compactification of complete, finite volume, irreducible, locally symmetric spaces of noncompact type. We will recall the reflection group trick and define these manifolds in Section 4. (Un)fortunately, these manifolds satisfy the Hopf conjecture. However, they have interesting properties: homotopy equivalences of such manifolds are homotopic to a homeomorphism; they can have infinite groups of self homotopy equivalences; and they form a new class of examples of spaces whose fundamental groups have the structure of a complex of groups.

We begin with a simple example. Let X be the product of two surfaces $S_1 \times S_2$, each of which has one boundary component ∂S_i . Then topologically, X is a manifold with boundary but smoothly, a manifold with corners. The boundary ∂X is the union of its two codimension 1 corners $S_1 \times \partial S_2$ and $\partial S_1 \times S_2$ along its codimension 2 corner $\partial S_1 \times \partial S_2$. If we double X along its boundary, we will get a manifold M with $\pi_2(M)$ nontrivial since there are noncontractible loops in the boundary that are contractible in the manifold. To get an aspherical manifold, we can proceed as follows.

Let D_3 be the dihedral group of order 6. Then D_3 acts on the Euclidean plane \mathbb{R}^2 as the group generated by reflections across two lines at angle $\pi/3$ with each other at the origin. A fundamental domain of D_3 is a sector D with angle $\pi/3$, which is a manifold with corners with

two codimension 1 corners and one codimension 2 corner. One can recover the plane (which is a manifold) by gluing 6 copies of D (which is a manifold with corners) in a natural way.

In the same way, we can glue six copies of the above X to get a closed manifold M on which D_3 acts with a fundamental domain a copy of X . The fundamental group of M has the structure of a hexagon of groups.

Some more interesting examples are the following. Given an arithmetic manifold, e.g. $\mathrm{SO}(3, \mathbb{R}) \backslash \mathrm{SL}(3, \mathbb{R}) / \Gamma$, for some torsion-free subgroup Γ of $\mathrm{SL}(3, \mathbb{Z})$, its Borel-Serre compactification is a manifold Y with corners. One can do the same thing as above with Y (instead of X) and an appropriate Coxeter group W (instead of D_3) to obtain an manifold M on which W acts with a fundamental domain a copy of Y . The manifold M is compact if and only if W is finite. If W is not finite, one can get a compact manifold by taking the quotient of M by a torsion-free, finite index subgroup of W . This is the general idea of the reflection group trick.

The above manifolds are examples of piecewise locally symmetric manifolds. The first main result of this paper is that they are aspherical.

Theorem 1 (Asphericity). *Piecewise locally symmetric manifolds are aspherical.*

One of the main steps in proving asphericity of piecewise locally symmetric manifolds is to prove that their fundamental groups are nonpositively curved complexes of groups. There have not been so many examples of nonpositively curved complexes of groups of dimension ≥ 2 . Piecewise locally symmetric manifolds provide many new examples of nonpositively curved complexes of groups.

(Non)rigidity. Since piecewise locally symmetric manifolds are made of pieces that are strongly rigid by Mostow-Prasad-Margulis rigidity, one natural question to ask is if or to what extent rigidity holds for this class of manifolds. In particular, we address the following question.

For M be a manifold. We denote by $\mathrm{Homeo}(M)$ the group of self-homeomorphisms of M and by $\mathrm{Homeo}_0(M)$ the group of self-homeomorphisms of M that are homotopic (not necessarily isotopic) to the identity map. Hence, $\mathrm{Homeo}(M) / \mathrm{Homeo}_0(M)$ is the group of self-homeomorphisms of M up to homotopy. The action of a homeomorphism on $\pi_1(M)$ induces a natural homomorphism

$$(1) \quad \eta: \mathrm{Homeo}(M) / \mathrm{Homeo}_0(M) \longrightarrow \mathrm{Out}(\pi_1(M)).$$

If M is aspherical, then η is always an injection since $\mathrm{Out}(\pi_1(M))$ is canonically isomorphic to the group of self homotopy equivalences of M up to homotopy. One can ask if η is a surjection, i.e. if a homotopy equivalence of M is homotopic to a homeomorphism.

It is known for a number of classes of manifolds that η is an isomorphism. These include closed surfaces, by the Dehn-Nielsen-Baer Theorem; infra-nilmanifolds, by Auslander [1]; finite-volume, complete, irreducible, locally symmetric, nonpositively curved manifold of dimension greater than 2, by Mostow Rigidity; closed, nonpositively curved manifolds of dimension greater than 4, by Farrell and Jones [8], and for solvmanifolds, by work of Mostow [13].

Theorem 2 (Rigidity/Nielsen realization). *Let M be a piecewise locally symmetric manifold of dimension $n > 2$. Assume that the pieces of M are irreducible. Then any homotopy equivalence of M is homotopic to a homeomorphism. Moreover, $\mathrm{Out}(\pi_1(M))$ can*

be realized as a group of homeomorphisms. That is, there is an injective homomorphism $\rho: \text{Out}(\pi_1(M)) \longrightarrow \text{Homeo}(M)$ such that

$$\begin{array}{ccc} & & \text{Homeo}(M) \\ & \nearrow \rho & \downarrow p \\ \text{Out}(\pi_1(M)) & \xleftarrow{\eta} & \text{Homeo}(M)/\text{Homeo}_0(M) \end{array} .$$

That $\text{Out}(\pi_1(M))$ can be realized as group of homeomorphisms of M means that there is a global solution to the Nielsen Realization problem. In general, one does not expect such a global solution. Morita [12] proved that if M is a surface of genus greater than 17, the group $\text{Out}(\pi_1(M))$ does not lift to $\text{Homeo}(M)$.

In proving Theorem 2, we obtain the following rigidity property of the decomposition of M into locally symmetric pieces.

Theorem 3. *Let M be a piecewise locally symmetric manifold of dimension $n > 2$, and let M_i , $i \in I$, be the locally symmetric pieces in the decomposition of M . Suppose that M_i are irreducible and have \mathbb{Q} -rank > 1 . Let $f: M_i \longrightarrow M$ be a π_1 -injective map. Then f is homotopic to a map $g: M_i \longrightarrow M$ that is a diffeomorphism onto a piece $M_j \subset M$.*

We believe that the above theorem is true for the case where the pieces have \mathbb{Q} -rank 1, but the technique we use to prove the Theorem 3 does not apply to the \mathbb{Q} -rank 1 case. The key in the proof of this theorem is to use nonpositively curved complexes of groups and apply a theorem of Farb in [7] on generation of lattices with \mathbb{Q} -rank ≥ 2 by certain nilpotent subgroups. In some way, the mechanism for rigidity is the same as for asphericity, i.e. both are governed by the nonpositively curved complex (of smaller dimension than M) given by the complex of groups structure of $\pi_1(M)$.

Theorem 3 means that the embedding of each piece M_i in M is rather rigid up to homotopy. It follows that the decomposition of M into locally symmetric pieces is unique. In other words, up to homotopy there is only one way to tile M by locally symmetric pieces, and that any homotopy equivalence of M must preserve the decomposition. This is actually the first step in proving Theorem 2. Pushing to homeomorphisms is the other step.

Despite the rigidity that Theorem 3 seems to suggest, there is a nonrigidity aspect of piecewise locally symmetric manifolds that appears in the structure of $\text{Out}(\pi_1(M))$. The group $\text{Out}(\pi_1(M))$ contains a (possibly trivial) free abelian normal subgroup $\mathcal{T}(M)$ whose non-identity elements are called *twists* (see below for the exact definition). These are analogous to Dehn twists in surface topology. It turns out that there are examples where $\mathcal{T}(M)$ is nontrivial. Thus, $\text{Out}(\pi_1(M))$ can be infinite, unlike the locally symmetric case, for which the outer automorphism group of the fundamental group is finite.

Theorem 4 (Computation of $\text{Out}(\pi_1(M))$). *Let M be a piecewise locally symmetric manifold and the pieces in the decomposition of M are irreducible locally symmetric manifolds. Then the outer automorphism group $\text{Out}(\pi_1(M))$ is an extension of a free abelian group $\mathcal{T}(M)$ by a group $\mathcal{A}(M)$, i.e. the following sequence is exact.*

$$1 \longrightarrow \mathcal{T}(M) \longrightarrow \text{Out}(\pi_1(M)) \longrightarrow \mathcal{A}(M) \longrightarrow 1.$$

If M has finitely many pieces, then $\mathcal{T}(M)$ is finitely generated and $\mathcal{A}(M)$ is finite.

Even though Nielsen realization holds for piecewise locally symmetric manifolds, not every such manifold admits a Riemannian metric such that $\text{Out}(\pi_1(M)) \cong \text{Isom}(M)$. This is in contrast to the case of Mostow-Prasad-Margulis rigidity.

Theorem 5. *Let M be a closed, piecewise locally symmetric manifolds of dimension ≥ 3 . If $\text{Out}(\pi_1(M))$ is infinite, then M does not admit any metric such that $\text{Out}(\pi_1(M))$ can be realized as a group of isometries.*

This paper is organized as follows. We recall the Borel-Serre compactifications of locally symmetric spaces in Section 2, and the reflection group trick in Section 3. We then define piecewise locally symmetric manifolds in Section 4. We remark on the lack of $\text{CAT}(0)$ metrics on piecewise locally symmetric manifolds, and raise a question in Section 5. The complex of groups structure of $\pi_1(M)$ is discussed in Section 6. We prove Theorem 1 in Section 7, and Theorem 3 in Section 8. We discuss the structure of $\text{Out}(\pi_1(M))$ and prove Theorem 4 in Section 9. Finally we prove Theorem 2 in Section 10. There is an appendix at the end of the paper that is a summary of theory of complexes of groups that we will need to use in this paper.

Acknowledgement. I would like to thank my advisor, Benson Farb, for his guidance, directions, constant encouragement, and extensive comments on earlier versions of this paper. I would like to thank Ilya Gekhtman and Dave Witte-Morris for a lot of help on the theory of arithmetic lattices. I would also like to thank Mike Davis and Shmuel Weinberger for useful conversations.

From now on, by a *locally symmetric* manifold we will mean a complete, finite volume, locally symmetric, nonpositively curved manifold of noncompact type.

2. COMPACTIFICATIONS OF LOCALLY SYMMETRIC SPACES

Let M be a connected, noncompact, finite-volume, complete, locally symmetric, nonpositively curved manifold of noncompact type. Then $M = \Gamma \backslash G/K$ for some semisimple Lie group G with K a maximal compact subgroup of G and Γ is a torsion-free lattice of G that is isomorphic to the fundamental group $\pi_1(M)$.

If M is an arithmetic manifold, then it has a compactification \overline{M} called the Borel-Serre compactification of M (see [2]). The space \overline{M} is a compact manifold with corners whose interior is diffeomorphic to M . The corners of \overline{M} correspond to Γ -conjugacy classes of rational parabolic subgroups of G . For example, let Γ be a torsion-free finite index subgroup of $\text{SL}(n, \mathbb{Z})$, and let

$$M = \text{SO}(n) \backslash \text{SL}(n, \mathbb{R}) / \Gamma.$$

Then M is a noncompact, finite volume, locally symmetric manifold. Consider the case where $n = 3$. The compactification \overline{X} is a manifold with corners with codimension 1 and codimension 2 strata. (See also Figure 2 for an schematic picture of \overline{M} .)

Recall that the parabolic subgroups of $\text{SL}(n, \mathbb{R})$ are precisely the subgroups that preserve a flag. There are two kinds of codimension 1 strata, which correspond respectively to the Γ -conjugacy classes of the following two parabolic subgroups of $\text{SL}(3, \mathbb{R})$.

$$P_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad P_2 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

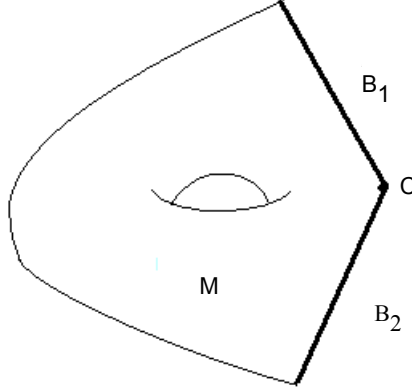


FIGURE 1. Schematic of a manifold with corners with two codimension 1 strata and one codimension 2 stratum.

and one kind of codimension 2 stratum, which corresponds to the Γ -conjugacy class of the parabolic subgroup

$$Q = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

For each $i = 1, 2$, the codimension 1 strata are the following spaces

$$B_i = (P_i \cap \mathrm{SO}(3)) \backslash P_i / (P_i \cap \Gamma),$$

which is a 2-torus bundle over $\mathrm{SO}(2) \backslash \mathrm{SL}(2, \mathbb{R}) / (\Gamma \cap \mathrm{SL}(2, \mathbb{Z}))$. The codimension 2 strata are of the form

$$C = (Q \cap \mathrm{SO}(3)) \backslash Q / (Q \cap \Gamma),$$

which is a compact nilmanifold.

In general, the Borel-Serre compactification of an arithmetic manifold M is constructed as follows. As before, $M = K \backslash G / \Gamma$ for some semisimple, linear, connected algebraic group defined over \mathbb{Q} with a maximal compact subgroup K , and Γ is an arithmetic lattice of G . Let $X = G / K$, which is a symmetric space of noncompact type and the universal cover of M .

Let P be a rational parabolic subgroup of G . Let N_P be the unipotent radical of P . The Langlands decomposition of P is $P = M_P A_P N_P$, where $N = N_P$, A_P is a \mathbb{Q} split torus, and M_P is a reductive subgroup of P with finitely many components. The groups M_P , A_P and N_P

are defined over \mathbb{Q} , and $M_P A_P = C_G(A_P)$, the centralizer of A_P in G . The latter fact together with N_P being normal in P implies that A_P normalizes $M_P N_P$.

Each parabolic P acts transitively on X . The Langlands decomposition of P gives X a *horospherical decomposition*

$$X = N_P \times A_P \times X_P,$$

where X_P is the symmetric space corresponding to M_P , i.e. $X = M_P/(M_P \cap K)$. In this case A_P is isomorphic to the multiplicative group $(0, \infty)^k$, for some positive integer k . The Borel-Serre partial compactification is

$$\overline{X} = \left(X \coprod_P X(P) \right) / \sim,$$

where P runs through all rational parabolic subgroups of G , and \sim is an equivalence relation defined as follows. For each pair of rational parabolic subgroups P and Q , let R be the smallest rational parabolic subgroup containing both P and Q . Such R always exists since G is a parabolic subgroup containing P and Q . There are embeddings ([10, Proposition III.5.7])

$$i_1: X_R \longrightarrow X_P, \quad \text{and} \quad i_2: X_R \longrightarrow X_Q.$$

Then for any $x \in X_R$, the points $i_1(x) \in X_P$ and $i_2(x) \in X_Q$ are defined to be equivalent.

The space \overline{X} has the structure of an analytic manifold with corners. For $g \in G_{\mathbb{Q}}$, the natural action of g on X extends to \overline{X} ([10, Proposition III.5.13]). The quotient $\Gamma \backslash \overline{X}$ is a compact analytic manifold with corners, called the *Borel-Serre compactification* of $\Gamma \backslash \overline{X}$, which we will denote \overline{M} . The manifold with corners \overline{M} is compact with interior diffeomorphic to M .

By the Margulis arithmeticity theorem [18, Theorem 6.1.2], if the Lie group G has \mathbb{R} -rank greater than 1, then any irreducible lattice of G is arithmetic. Hence, if M is higher rank and irreducible, then M always has a compactification that is a manifold with corners.

If the \mathbb{R} -rank of M is 1, then M may not be arithmetic, in which case M is negatively curved with pinched curvature. Then M has finitely many ends, or *cusps*. Each cusp is diffeomorphic to $[0, \infty) \times S$ for some compact $(n-1)$ -dim manifold S . The fundamental group of each cusp, or each *cusp subgroup*, corresponds to a maximal subgroup of $\pi_1(M)$ of parabolic isometries fixing a point on the boundary at infinity of \widetilde{M} . The parametrization $[0, \infty) \times S$ of a cusp can be taken so that each cross section $a \times S$ is the quotient of a horosphere in \widetilde{M} by the corresponding cusp subgroup. If we delete the $(b, \infty) \times S$ part of each cusp $[0, \infty) \times S$ of M , the resulting space is a compact manifold with boundary which we call \overline{M} . We can identify M with the interior of \overline{M} . Hence, \overline{M} is a compactification of M . We choose b large enough so that the boundary components $b \times S$ of different cusps do not intersect.

If M is reducible, i.e. M is finitely covered by a product $Y = \prod_{j \in J} Y_j$, for some finite collection of noncompact, irreducible, locally symmetric spaces Y_j , then M has an obvious compactification as follows. The cover Y has a compactification $\overline{Y} = \prod_{j \in J} \overline{Y}_j$. Since the cover space action on Y extends to \overline{Y} , this gives a compactification of M as the quotient of \overline{Y} by the covering space action.

3. THE REFLECTION GROUP TRICK

In [6], Davis constructs, for an appropriate Coxeter system (W, S) and a mirrored space X , a space $U(W, X)$ with a proper W -action. We give a brief overview of this construction. (See [6] for more detailed discussion.)

A *mirrored space* X over S is a topological space X with a collection of closed subspaces $\{X_s \mid s \in S\}$ for some index set S . For each $x \in X$, let

$$S(x) = \{s \in S \mid x \in X_s\}.$$

The *nerve* $N(X)$ of X is a simplicial complex defined as follows. The vertex set of $N(X)$ is the set of $s \in S$. A nonempty subset T of S is a simplex of $N(X)$ if and only if X_T is nonempty.

Let (W, S) be a Coxeter system, i.e. W has the following presentation.

$$W = \langle s_i \in S \mid (s_i)^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle.$$

for some (symmetric with positive integer entry) Coxeter matrix m_{ij} . For each subset $T \subseteq S$, define W_T to be the subgroup of W that is generated by all $s \in T$. The *nerve* a Coxeter system (W, S) is denoted by $L(W, S)$ and is simplicial complex whose vertices correspond to $s \in S$, and $T \subset S$ is simplex if T *spherical subset* of S . (A spherical subset of S is a subset whose elements generate a finite subgroup.)

Define an equivalence relation on $W \times X$ as $(h, x) \sim (g, y)$ if $x = y$ and $h^{-1}g \in W_{S(x)}$. Then $U(W, X)$ is defined to be the quotient space $(W \times X)/\sim$. It is easy to check that W acts on $U(W, X)$ with quotient X . This action is proper if X is Hausdorff and the mirror structure of X is *W-finite*, that is, for each $T \subset S$ such that W_T is infinite, the intersection $\bigcap_{s \in T} X_s$ is empty.

One can think of the space $U(W, X)$ as what they see if they are standing in a room of shape X with real mirrors X_s 's. For example, an observer in a cubical room with one mirror on a wall will think that they are in a doubly larger room that is the union of two identical cubical rooms along the wall with the mirror. This corresponds to the case where X is a cube with one of the faces a mirror and W is a one-generator Coxeter system. Another example is when a 2-dimensional observer is in a triangular room with angles π/m , π/n and π/k for some integers $m, n, k > 1$. In the case where $m = n = k = 3$, for example, the observer will think that they are in a flat plane tiled with copies of the room (by looking at the repetition of the same furniture in the room for example). This corresponds to the case where X is an equilateral triangle with mirrors on its three edges and W is the $(3, 3, 3)$ triangle group.

Manifolds with corners: A natural class of mirrored spaces is the class of *nice manifolds with corners*. A *smooth n -manifold with corners* X is a second countable Hausdorff space that is differentiably modelled on the standard $\mathbb{R}_+^n = [0, \infty)^n$. Given a point $x \in X$ and chart ϕ on a neighborhood of x , the number $c(x)$ of coordinates of $\phi(x)$ is independent of the chart ϕ . For $0 \leq k \leq n$, a connected component of $c^{-1}(k)$ is a *stratum* of X of codimension k . The closure of a stratum is *closed stratum*. For each $x \in X$, let $m(x)$ be the set of closed codimension 1 strata containing x . A manifold with corners X is *nice* if the cardinality of $m(x)$ is 2 for all x with $c(x) = 2$.

A nice manifold with corners X has a natural mirror structure where each closed codimension 1 stratum is a mirror. If (W, S) is a Coxeter system and X is a nice manifold with corners whose codimension 1 strata are C_s for $s \in S$ gives X a mirror structure that is *W-finite*, then

$U(W, X)$ is a manifold, and W acts on $U(W, X)$ properly and *locally linearly* ([6, Proposition 10.1.10]).

If X is a nice manifold with corners with the natural mirror structure, then the nerve of X is a *convex cell complex*. There is a Coxeter system (W, S) such that $L(W, S)$ is the barycentric subdivision of the nerve $N(X)$ of X . For example, for any convex cell complex Λ , there is a right-angled Coxeter system (W, S) with nerve $b\Lambda$ ([6, Lemma 7.2.2]). For such a Coxeter system, the space $U(W, X)$ is a manifold. One just needs to check that the mirror structure of X is W -finite. The space $U(W, X)$ is a differential manifold with differential structure induced by those on each copies of X in $U(W, X)$.

The space $U(W, X)$ need not be compact even if X is compact since W can be infinite. However, it has a compact manifold quotient since W acts on $U(W, X)$ properly with compact quotient, and W has a finite index torsion-free subgroup Γ . The latter is because W admits a faithful representation into some general linear group. Let M be the quotient of $U(W, \bar{X})$ by Γ . Then M is a compact manifold.

4. PIECEWISE LOCALLY SYMMETRIC SPACES: DEFINITION AND EXAMPLES

Let X be a locally symmetric manifold and let \bar{X} be the Borel-Serre compactification of X . Then \bar{X} is a nice manifold with corners ([10, Proposition III.5.14]).

Let (W, S) be a Coxeter system with nerve $L(W, S)$ the barycentric subdivision of the nerve $N(\bar{X})$ of \bar{X} . Then $U(W, \bar{X})$ is a smooth manifold with the smooth structure induced by that of \bar{X} . There is a natural action of W on $U(W, \bar{X})$. We call manifolds that are quotients of $U(W, \bar{X})$ by subgroups of W *piecewise locally symmetric spaces*. These manifolds need not be compact, but there are always compact ones, e.g. $U(W, \bar{X})/\Gamma$, for some torsion-free finite index subgroups of W .

We remark that one can similarly define *piecewise locally symmetric manifolds with boundary* by picking W to be a Coxeter group that is generated by a proper subset of the set of mirrors of \bar{X} .

Now, we give a few examples of piecewise locally symmetric spaces.

- a) The simplest example of a piecewise locally symmetric space is the double of a non-positively curved, locally symmetric space X with one cusp. In this case, the Coxeter group W is that with one generator since X has only one boundary component.
- b) Let X be the product of two \mathbb{R} -rank 1 locally symmetric spaces X_1 and X_2 , each of which has one cusp. The boundary ∂X is the union of $B_1 = X_1 \times \partial X_2$ and $B_2 = \partial X_2 \times X_1$ along the corner $C = \partial X_1 \times \partial X_2$. So X has two codimension 1 strata B_1 and B_2 , which are the mirrors of X , and one codimension 2 stratum C . Let W be the Coxeter system generated by B_1 and B_2 with the relation $(B_1 B_2)^k = 1$ for some $k > 1$. That is, W is the dihedral group D_k . Let M be $U(W, X)$. Then M is a piecewise locally symmetric space. Since W is finite, M is a compact manifold. As we will see in Section 6, the fundamental group of M has the structure of a $2n$ -gon of groups. (See [4], [16], or the appendix below for the theory of complexes of groups).
- c) If the spaces X_1 and X_2 in the above example have more than one cusp, i.e. the compactification \bar{X}_1 or \bar{X}_2 has more than one boundary component, then the group W is in general not the reflection group of a polygon. The fundamental group of M has the structure of a 2-dimension complex of groups.

- d) Let Γ be a torsion-free, finite index subgroup of $\mathrm{SL}(n, \mathbb{Z})$ and let X be $\mathrm{SO}(n) \backslash \mathrm{SL}(n, \mathbb{R}) / \Gamma$. The Borel-Serre compactification \overline{X} is a manifold with corners as we saw in section 2. Since Γ is a subgroup of $\mathrm{SL}(n, \mathbb{Z})$, the set of mirrors of X may contain more than $(n-1)$ but finitely many mirrors. Let W be the corresponding right angle Coxeter system. The manifold $U(W, X)$ is the union of copies of X glued to each other along pairs of codimension 1 boundary strata.

Each copy of \overline{X} is called a *fundamental chamber* of M . Note that all the pieces M_i are diffeomorphic. For each piece M_i of M , there is a retraction of $r: M \rightarrow M_i \cong \overline{X}$ defined as follows. For each $x \in M$, let \hat{x} be a lift of x in $U(\overline{X}, W)$. Then $r(x) = p(\hat{x})$, where $p: U(W, \overline{X}) \rightarrow \overline{X}$ is the covering projection.

Two codimension 1 strata S_1 and S_2 is said to have the same *type* if $r(S_1) = r(S_2)$.

5. LACK OF LOCALLY CAT(0) METRICS FOR PIECEWISE LOCALLY SYMMETRIC MANIFOLDS

Firstly we recall the definition of a CAT(κ) metric space ([4]). Let X be a metric space. Let M_κ^2 be the complete, simply connected, 2-dimensional manifold with constant curvature equal to κ . Let D_κ be equal to $\pi/\sqrt{\kappa}$ if $\kappa > 0$, and ∞ otherwise.

Let \triangle be a geodesic triangle in X with vertices p, q, r , and with perimeter less than $2D_\kappa$. A triangle $\overline{\triangle}$ in M_κ^2 with vertices $\bar{p}, \bar{q}, \bar{r}$ is a comparison triangle for \triangle if

$$d(p, q) = d(\bar{p}, \bar{q}), \quad d(q, r) = d(\bar{q}, \bar{r}), \quad \text{and} \quad d(r, p) = d(\bar{r}, \bar{p}).$$

A point \bar{x} on the geodesic segment $[\bar{p}, \bar{q}]$ is a comparison point for $x \in [p, q]$ if $d(p, x) = d(\bar{p}, \bar{x})$. Comparison points on $[\bar{q}, \bar{r}]$, and $[\bar{r}, \bar{p}]$ are defined similarly.

Such a triangle \triangle is said to satisfy the CAT(κ) inequality if for all $x, y \in \triangle$, and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

A geodesic metric space X is CAT(κ) if all such triangles satisfy the CAT(κ) inequality. A metric space X is said have *curvature* $\leq \kappa$ if it is locally a CAT(κ) space.

It is a well-known fact that if a Riemannian manifold M has sectional curvature ≤ 0 , then M is locally CAT(0).

Piecewise locally symmetric manifolds generally do not admit a locally CAT(0) metric. For example, if a closed piecewise locally symmetric M is obtained from complex hyperbolic pieces, then M does not admit a locally CAT(0) metric. More generally, if M is obtain from any locally symmetric manifold whose fundamental group has a solvable, nonabelian subgroup, then M does not admit a locally CAT(0) metric. This is because of Theorem 7.8 in [4], which says that any solvable subgroup of a locally CAT(0) group $\pi_1(M)$ must be virtually abelian.

However, some piecewise locally symmetric manifold do admit nonpositively curved Riemannian metrics (and thus, locally CAT(0) metrics). For example, if a piecewise locally symmetric M is obtained from hyperbolic pieces, then one can smooth out the hyperbolic metric (which is singular at gluing) to get a nonpositively curved Riemannian metric. Another example if the manifold in first example of the introduction. One can take the product of the hyperbolic metric on S_1 and S_2 . Gluing these metrics will give a locally CAT(0), singular Riemannian metric on M . Nonetheless, one can smooth out this metric to get a nonpositively curved metric on M .

If M is obtained from pieces that are products of at least 3 hyperbolic manifolds with cusps, then M has a locally CAT(0) metric, which can be taken to be the glued up product hyperbolic metrics from each piece.

Question: If M is as in the previous line, then does M admit a nonpositively curved C^2 (respectively, smooth) Riemannian metric?

6. FUNDAMENTAL GROUPS AND UNIVERSAL COVERS OF PIECEWISE LOCALLY SYMMETRIC MANIFOLDS

Let M be a piecewise locally symmetric manifold of \mathbb{R} -rank ≥ 2 , and let $\{M_i\}_{i \in I}$ be the pieces in the decomposition of M . That is, there is a locally symmetric space X with finite volume, a Coxeter system (W, S) and a finite index, torsion-free subgroup W' of W such that $U(W, \bar{X})$ is a manifold and $M = U(W, \bar{X})/W'$. Then space $U(W, X)$ is a complex of spaces over some complex Λ (see [5] for the definition of a complex of spaces). The barycentric subdivision of Λ is the complex $\Sigma := \Sigma(W, S)$ associated to (W, S) as defined in [6]. We briefly recall the definition of Σ . Let K be the cone on the barycentric division of $L(W, S)$. Then $\Sigma = U(W, K)$. Some important properties of Σ are as follows.

There is a natural cell structure on Σ ([6]) so that

- a) its vertex set of W , its 1-skeleton is the Cayley graph of (W, S) , and its 2-skeleton is a Cayley 2-complex,
- b) each cell is a Coxeter polytope,
- c) the link of each vertex is isomorphic to $L(W, S)$,
- d) a subset of W is the vertex set of a cell if and only if it is a spherical coset of (W, S) .

The space $U(W, \bar{X})$ has the structure of a complex of spaces, where the base complex is Σ with the cell structure as in the above theorem. (See the appendix below for a short summary of necessary results from the theory of complex of groups and spaces.) The barycentric subdivision of this cell structure is $U(W, K)$. Since W' acts on $U(W, \bar{X})$ preserving fibers, the quotient $M = U(W, \bar{X})/W'$ has the structure of a complex Y_M of spaces, for $Y_M = \Lambda/W'$ since W' is torsion-free. The fiber over each point is a stratum of a copy of \bar{X} .

The complex of spaces structure of M gives $\pi_1(M)$ the structure of the fundamental group of a complex of groups $G(T_M)$ over T_M (see the appendix of this paper or [4] and [11] for complexes of groups). The vertex groups of $G(T_M)$ are the fundamental groups of the pieces M_i . The edge groups of $G(T_M)$ are the fundamental groups of the each codimension 1 stratum of each piece. The k -dimensional cell groups are the fundamental groups of each codimension k stratum. The injective homomorphism from k -dimensional cell group to a $(k - 1)$ -dimensional cell group are the obvious inclusion of the corresponding strata.

7. DEVELOPABILITY OF THE FUNDAMENTAL GROUP AND ASPHERICITY

In general, the vertex groups are not subgroups of a complex of groups [16]. Complexes of groups that contain their vertex groups as subgroups are called *developable*. A necessary condition for a complex of groups to be developable is that it is *nonpositively curved* [4].

Theorem 6 (Developability of π_1). *Let M be a piecewise locally symmetric space. Then the natural complex of groups structure of $\pi_1(M)$ is nonpositively curved and thus is developable.*

Proof. Let M_i , $i \in I$, be the pieces in the decomposition of M . In order to prove that $\pi_1(M)$ is nonpositively curved as a complex T_M of groups, we need to give T_M a metric such that the local development (see [4] or the appendix about local developments) of each vertex in T_M is nonpositively curved. We give T_M a piecewise Euclidean metric as follows.

Firstly, give $\Sigma(W, S)$ a piecewise Euclidean metric as in [6, Section 12.1]. Since T_M is a quotient of Σ by a group of isometries, this metric induces a piecewise Euclidean metric on the polyhedral complex T_M . Now consider the barycentric subdivision of T_M . There are only finitely many isometry classes of simplices of T_M . So the local development $\text{st}(\tilde{\sigma})$ of each vertex σ of T_M is a piecewise Euclidean complex with finite shape. Given this, Theorem 5.2 in [4] says that $\text{st}(\tilde{\sigma})$ is CAT(0) if the link at each vertex of $\text{st}(\tilde{\sigma})$ is CAT(1).

For each i , let B_i be the rational spherical Tits building of X_i . Let σ be a vertex of T_M that corresponds to some X_i . Then the local development $\text{st}(\tilde{\sigma})$ of σ is isomorphic to the cone on B_i as a simplicial complex, with cone point $\tilde{\sigma}$.

The complex $\text{Lk}(\tilde{\sigma}, \text{st}(\tilde{\sigma}))$ is piecewise spherical and has simplices of size $\geq \pi/2$, i.e. each edge of a simplex has length $\geq \pi/2$ ([6, Lemma 12.3.1]. By a theorem of Moussong ([6, Lemma I.7.4]), B_i is CAT(1) if and only if it is a metric flag complex (see [6, Definition I.7.1] for the definition of metric flag complexes). But $\text{Lk}(\tilde{\sigma}, \text{st}(\tilde{\sigma}))$, which is isomorphic to B_i , is a flag complex. Each simplex of B_i corresponds to a spherical subgroup of W . For the same reason as in [6, Proof of Lemma 12.3.1], the piecewise spherical complex B_i is a metric flag complex and thus CAT(1). Hence $\text{st}(\tilde{\sigma})$ is CAT(0).

Let ζ be a vertex of T_M that corresponds to the barycenter of a (polyhedral) cell of positive dimension. Let σ be a vertex of this cell. Then $\text{st}(\tilde{\zeta})$ is isometric to a neighborhood of point p in $\text{st}(\tilde{\sigma})$. We can pick p to be in the interior of a cell of type ζ in $\text{st}(\tilde{\sigma})$. Since $\text{st}(\tilde{\sigma})$ is CAT(0) as shown above, $\text{st}(\tilde{\zeta})$ is CAT(0).

We have checked that the local development of each vertex in the barycentric subdivision of T_M is CAT(0). Therefore, the complex of group of $\pi_1(M)$ is nonpositively curved and thus, developable. □

Lemma 7. *The fundamental group $\pi_1(M)$ of a piecewise locally symmetric space M has trivial center.*

Proof. As before, the universal cover \tilde{M} is a complex \tilde{T} of spaces. Consider the action of $\pi_1(M)$ on \tilde{T} . If $g \in \pi_1(M)$ is in the center of $\pi_1(M)$, then g preserves the set of common fixed points in \tilde{T} of each vertex group of $\pi_1(M)$ (which contains exactly just the corresponding vertex). Hence, g fixes every vertex of \tilde{T} . Thus, $g = 1$ and $\pi_1(M)$ has trivial center. □

Now we prove Theorem 1.

Proof of Theorem 1. Since the complex of groups structure of $\pi_1(M)$ is developable, the universal cover \tilde{M} is a complex of \tilde{M}_i , which is contractible. Hence, \tilde{M} is homotopy equivalent to its nerve, which is CAT(0) and thus contractible. Therefore, \tilde{M} is contractible. □

8. PROOF OF THEOREM 3 AND AUTOMORPHISMS OF FUNDAMENTAL GROUPS

Let M be a piecewise locally symmetric manifold that satisfies the conditions of Theorem 3, and let $\{M_i\}_{i \in I}$ be the pieces in the decomposition of M into locally symmetric pieces.

Since each piece M_i of M is aspherical, we only need to prove the statement on the level of fundamental group. That is, it suffices to prove that following lemma.

Lemma 8. *Let G_v be the fundamental group of an irreducible locally symmetric manifold Y of \mathbb{Q} -rank > 1 . Let $\phi: G_v \rightarrow \pi_1(M)$ be an injective homomorphism. Then $\phi(G_v)$ is a conjugate of a vertex group of $\pi_1(M)$.*

Before proving the lemma, we state three results that we are going to use in the proof of the lemma.

Theorem 9 ([7]). *Let N be a finitely generated, torsion-free, nilpotent group acting on a CAT(0) space Y by semi-simple isometries. Then either N has a fixed point or there is an N -invariant flat L on which N acts by translations and hence, factoring through an abelian group.*

Corollary 10 ([7]). *Let N be a finitely generated, torsion-free, nilpotent group acting on a CAT(0) space Y by semi-simple isometries. Then*

- 1) *If $g^m \in [N, N]$ for some $m > 0$, then g has a fixed point.*
- 2) *If N is generated by elements each of which has a common fixed point, then N has a global fixed point.*

The following theorem is due Farb ([7]) and was proved in a more general setting with \mathbb{Q} replaced by an algebraic number field k .

Theorem 11 ([7]). *Let Γ be an irreducible lattice of \mathbb{Q} rank $r \geq 2$, and let Γ act on a CAT(0) space Y . Then exists a collection of subgroups $\mathcal{C} = \{\Gamma_1, \Gamma_2, \dots, \Gamma_{r+1}\}$ such that*

- (1) *The groups in \mathcal{C} generate a finite index subgroup of Γ .*
- (2) *Any proper subset of \mathcal{C} generates a nilpotent subgroup U of Γ .*
- (3) *There exists $m \in \mathbb{Z}^+$ so that for each $\Gamma_i \in \mathcal{C}$, there is a nilpotent group $N < \Gamma_i$ so that $r^m \in [N, N]$ for all $r \in \Gamma_i$.*

As pointed out in [7], it follows from the corollary above that for each such U , the set $\text{Fix}(U)$ in Y is nonempty.

Lemma 12. *Let Z be a CAT(1) piecewise spherical simplicial complex in which all cells have size $\geq \pi/2$. Let X, Y be cells of Z , and let $x \in X$, $y \in Y$. If $d(x, y) < \pi/2$, then $X \cap Y \neq \emptyset$, and there is a point $p \in X \cap Y$ such that $\angle_p(x, y) < \pi/2$.*

Proof. Suppose that $X \cap Y = \emptyset$. Let γ be a unit speed geodesic connecting x and y so that $\gamma(0) = x$ and $y \in \gamma([0, \pi/2])$. Let $a < 0$ and $b > 0$ so that $\gamma([a, b])$ is a connected component of the intersection of γ with X . Let v be a vertex of X , and let A_v be the union of all cells containing v . Then $X \subset A_v$. We pick v so that $d(v, \gamma(a)) > d(v, \gamma(b))$.

Let B_v be the closed $\pi/2$ -ball centered at v . Then B_v is isometric to the spherical cone $\text{Cone}(\text{Lk}(v, Z))$ on the link of v in Z . If γ passes through v , then $d(v, y) < \pi/2$. Hence $v \in Y$, and thus $v \in X \cap Y$, which contradicts the above assumption. So γ does not pass through v .

For each t such that $\gamma(t) \in A_v$, let s_t be the geodesic segment connecting v passing through $\gamma(t)$ with length $\max(d(v, \gamma(t)), \pi/2)$. Let S be the surface defined as the union of all such s_t . In the same way as in [6, Proof of Lemma I.6.4], the surface S is a union of triangles with common vertex v glued together in succession along γ . Thus we can develop an S along γ

locally isometric to \mathbb{S}^2 , i.e. there is a map $f: S \rightarrow \mathbb{S}^2$ that is a local isometry such that $f(v)$ is the North pole. Hence, $f(\gamma)$ is a geodesic in \mathbb{S}^2 that misses $f(v)$. Also, $f(S)$ contains the Northern hemisphere of \mathbb{S}^2 .

The image $f(\gamma)$ cuts inside the region $f(S)$, which contains the Northern hemisphere N of \mathbb{S}^2 , so $f(\gamma)$ has length $\geq \pi/2$. Let d be such that $f(\gamma(d))$ is where $f(\gamma)$ exits $f(S)$. It is not hard to see that $\pi/2 > d > b > 0$. Since $d(v, \gamma(a)) > d(v, \gamma(b))$, it follows that $d(f(v), f(\gamma(a))) > d(f(v), f(\gamma(b)))$. Hence $d(f(\gamma(0)), f(\gamma(d))) > \pi/2$ by spherical geometry, which is a contradiction since γ has unit speed. Therefore, $X \cap Y \neq \emptyset$.

We can pick the point p to be v . That $\angle_p(x, y) < \pi/2$ also follows from spherical geometry. \square

Lemma 13. *Let Z be a CAT(1) piecewise spherical simplicial complex in which all cells have size $\geq \pi/2$. Let X, Y be cells of Z , and let $x \in X, y \in Y$. Let X' (respectively, Y') be a face of X (respectively, Y) whose interior contains x (respectively, y). If $d(x, y) < \pi/2$, then $X' \cap Y'$ contains face C such that $d(x, C) < \pi/2$.*

Proof. We induct on the dimension of Z . The base case is when Z has dimension 1, in which case the lemma is obvious. Suppose that Z has dimension > 1 . By Lemma 12, there is a point $p \in X \cap Y$ such that $\angle_p(x, y) < \pi/2$. Let u (respectively, v) be the intersection of the geodesic ray px (respectively, py) with the link $\text{Lk}(p, Z)$ of p in Z . Since $\angle_p(x, y) < \pi/2$, we have $d_{\text{Lk}(p, Z)}(u, v) < \pi/2$. Note that $\text{Lk}(p, Z)$ is also a CAT(1) piecewise spherical simplicial complex in which all cells have size $\geq \pi/2$.

Let U' (respectively, V') be a face in $\text{Lk}(p, Z)$ whose interior contains u (respectively, v). Apply the induction hypothesis, there is a point $q \in \text{Lk}(p, Z)$ such that $U' \cap V'$ contains face D such that $d(u, D) < \pi/2$.

If $d_Z(p, x) < \pi/2$, then let C be the face in Z that is spanned by p and D . Otherwise, let C be the simplex in Z that corresponds to C in $\text{Lk}(p, Z)$. Then C satisfies the condition in the lemma. \square

Now we prove Lemma 8.

Proof of Lemma 8. To show that $\phi(G_v)$ is a conjugate of a vertex group of $\pi_1(M)$, all we need to prove is that the action of $\phi(G_v)$ on the complex \tilde{T} fixes a vertex. Let M_k be a locally symmetric piece of M . By Theorem 11, the group $\Gamma = \pi_1(M_k)$ is virtually generated by a collection \mathcal{C} of nilpotent subgroups N_1, N_2, \dots, N_{r+1} . We only need Γ to be virtually generated by 3 nilpotent groups that satisfies the conclusion of Theorem 11. So we, instead, write $\mathcal{C} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$, for $\Gamma_1 = N_1, \Gamma_2 = N_2$, and Γ_3 is the (nilpotent) group generated by N_3, N_4, \dots, N_{r+1} . Observe that Γ_i 's satisfy the conclusion of Theorem 11. For each $i = 1, 2, 3$, let F_i be the set of points that is fixed by all elements of Γ_i .

$$F_i = \text{Fix}(\Gamma_i).$$

Let

$$W_{ij} = F_i \cap F_j,$$

for $i, j = 1, 2, 3$. By Theorem 11, each W_{ij} is nonempty. It is clear that F_i 's and W_{ij} 's are convex.

Suppose that

$$\cap_{i=1,2,3} F_i = \emptyset.$$

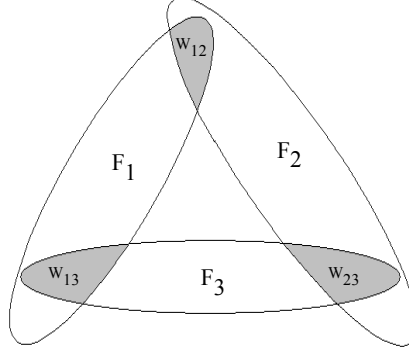


FIGURE 2. The Fix sets F_1 , F_2 and F_3 and their pairwise intersections.

Let $x_i \in W_{i3}$, for $i = 1, 2$, be such that the distance $d(x_1, x_2) = d(W_{13}, W_{23})$. Observe that x_i must lie on the boundary of F_i , for $i = 1, 2$ since x_1 and x_2 are points realizing the distance between two convex set W_{13} and W_{23} .

Let $y \in W_{12}$. The geodesic γ_{yx_1} (and γ_{yx_2} , respectively) lies in F_1 (and F_2 , respectively) since F_1 and F_2 are convex. Without loss of generality, suppose that (see [4] for the definition of *angle*)

$$\angle_{x_1}(y, x_2) < \pi/2,$$

We claim that there is a simplex S in W_{13} containing x_1 and the angle

$$\angle_{x_1}(S, x_1x_2) < \pi/2.$$

Given the claim, it follows that S contains some point x'_1 other than x_1 such that $d(x'_1, x_2) < d(x_1, x_2)$, which is a contradiction to the choice of x_1 and x_2 . Therefore,

$$\cap_{i=1,2,3} F_i \neq \emptyset.$$

That is, Γ has a finite index group Γ' that fixes a point. Let H be a finite index subgroup of Γ' that is normal in Γ . Then the fix set of H in L is nonempty. Since Γ acts on $\text{Fix}(H)$ with bounded orbit, it follows that Γ has a global fixed point.

Now we prove the claim. Let Q be the simplex whose interior contains x_1 .

Suppose that Q is a point. Then $Q = \{x_1\}$. Let L be the link of x_1 . Then L is a CAT(1) piecewise spherical simplicial complex. Let A (respectively, B) be the cell in \tilde{T} that intersects nontrivially with x_1y (respectively, x_1x_2). Let $A' = \text{Lk}(x_1, A)$ and $B' = \text{Lk}(x_1, B)$. Let $u = L \cap x_1y$ and $v = L \cap x_1x_2$. Then $d(u, v) < \pi/2$ in L . Let U (respectively, V) be a face of A' (respectively, B') whose interior contains u (respectively, v). By Lemma 13, there is a face $C \subset U \cap V$ such that $d_L(C, v) < \pi/2$. Let α (respectively, β) be the span of x_1 and U (respectively, V). Then $\alpha \subset F_1$ and $\beta \subset F_3$. Let S be the span of x_1 and C . Therefore, the edge $S \subset W_{13}$ and has angle $< \pi/2$ with x_1x_2 .

Suppose that Q is not a point. Then $L := \text{Lk}(x_1, \tilde{T})$ is the spherical join $\text{Lk}(x_1, Q) * \text{Lk}(Q, \tilde{T})$. If $\angle_{x_1}(x_1x_2, Q) < \pi/2$. Then pick $q \in Q$ such that $\angle_{x_1}(x_1x_2, x_1q) < \pi/2$. Since $Q \subset W_{13}$ (because x_1 is in the interior of Q and $x_1 \in W_{13}$). We can let S be the edge $x_1q \subset W_{13}$.

Suppose that $\angle_{x_1}(x_1x_2, Q) \geq \pi/2$ (in which case we have equality). Then x_1x_2 intersects nontrivially with $\text{Lk}(Q, \tilde{T})$ at v . If x_1y also intersects nontrivially with $\text{Lk}(Q, \tilde{T})$ at some point u , then argue as in the case Q is a point using the fact that $\text{Lk}(Q, \tilde{T})$ is CAT(1) piecewise spherical simplicial complex in which all cells have size $\geq \pi/2$ and applying Lemma 13.

Suppose that x_1y does not intersect nontrivially with $\text{Lk}(Q, \tilde{T})$. Then there is a point $q \in Q$ such that x_1y intersects with $H := q * \text{Lk}(Q, \tilde{T})$ nontrivially. Let $u = x_1y \cap H$ and let $v = x_1x_2 \cap H$. Then $d_H(q, v) = \pi/2$ and $d_H(u, v) < \pi/2$. Therefore, the angle $\angle_q(u, v) < \pi/2$. By applying Lemma 13 to the link $\text{Lk}(q, H) = \text{Lk}(Q, \tilde{T})$, we deduce that there is $C \subset \text{Lk}(Q, \tilde{T})$ such that the span S of Q and C is contained in W_{13} and $d_L(p, v) < \pi/2$. Thus $\angle_{x_1}(x_2, S) < \pi/2$.

□

The following theorem is an immediate corollary of Lemma 8 (for the case where \overline{M}_i has \mathbb{Q} -rank > 1). This theorem is true if we drop the \mathbb{Q} -rank > 1 condition. The case where the pieces M_i 's of M have \mathbb{Q} -rank 1 is dealt with in [15]. The case where M_i 's are negatively curved, locally symmetric manifolds is dealt with in [14]. So we state it with no assumptions on the \mathbb{Q} -rank or rational structure of M_i 's.

Theorem 14. *Let M be a piecewise locally symmetric space. Assume that each piece in the decomposition of M is irreducible. Let $\phi: \pi_1(M) \rightarrow \pi_1(M)$ be an automorphism. If G_v is a vertex group $\pi_1(M)$, then $\phi(G_v)$ is a conjugate of some vertex subgroup of M .*

Therefore, any automorphism of $\pi_1(M)$ preserves that complex of groups structure of $\pi_1(M)$ in the above sense.

9. $\text{Out}(\pi_1(M))$: TWISTS AND TURNS

If M is finite-volume, complete, irreducible, locally symmetric and nonpositively curved of noncompact type and of dimension > 2 , then by the Mostow Rigidity Theorem, $\text{Out}(\pi_1(M)) \cong \text{Isom}(M, g_{loc})$, where g_{loc} is the locally symmetric metric on M . This implies that $\text{Out}(\pi_1(M))$ is finite since $\text{Isom}(M)$ is finite by Bochner's theorem. One might expect that if M is piecewise locally symmetric, given the Theorem 3, the group $\text{Out}(\pi_1(M))$ will also be finite. However, this is not true, since there can be infinite order homeomorphisms that we call *twists*. (In the 2-dimensional case, these are Dehn twists in surface topology.)

To warm up, we give two examples where $\text{Out}(\pi_1(M))$ is infinite and what describe twists are in these cases.

9.1. Examples of infinite $\text{Out}(\pi_1(M))$.

9.1.1. *The pieces M_i 's have \mathbb{Q} -rank 1.* Firstly we consider the case of piecewise \mathbb{Q} -rank 1 manifolds. For example, let M be the double of a hyperbolic manifold N with one cusp. Let M_1 and M_2 be the two copies of N in M . Then $\pi_1(M) = G_1 *_C G_2$, where $G_i = \pi_1(M_i)$, for $i = 1, 2$, and C is the fundamental group of the cusp. Pick an element $c \neq 1$ in the center of C . Let $\phi: \pi_1(M) \rightarrow \pi_1(M)$ be induced by

$$\phi(g) = \begin{cases} g & \text{if } g \in G_1 \\ cgc^{-1} & \text{if } g \in G_2. \end{cases}$$

It is clear that ϕ extends to an automorphism of $G_1 *_C G_2$ since ϕ is an automorphism when restricted to G_1 and G_2 and agrees on the intersection of G_1 and G_2 .

Lemma 15. *Let M and ϕ be as above. Then ϕ is an infinite order element of $\text{Out}(\pi_1(M))$.*

Proof. For all $k \in \mathbb{N}$, ϕ^k is the identity on G_1 and conjugation by c_0^k on G_2 . Suppose ϕ^k is an inner automorphism of $G_1 *_C G_2$ for some $k \in \mathbb{N}$. Then there exists $g \in G_1 *_C G_2$ such that $\text{Conj}(g) \circ \phi$ is the identity on $G_1 *_C G_2$. This implies that for $g_1 \in G_1$, we have $g\phi^k(g_1)g^{-1} = g_1$. So $gg_1g^{-1} = g_1$ since ϕ^k is the identity on G_1 . Thus, g is in the centralizer of G_1 in $G_1 *_C G_2$.

We claim that g is in the centralizer of G_1 . Consider the action of $G_1 *_C G_2$ on the Bass Serre tree T of $G_1 *_C G_2$. The fixed set of G_1 is a vertex v . Since g commutes with every element of G_1 , the fixed set of G_1 must be preserved by g . Hence g belongs to the stabilizer of v , which is G_1 . Therefore, g is in the centralizer of G_1 and thus $g = 1$. This implies that ϕ is the identity homomorphism, which is a contradiction since ϕ acts nontrivially on G_2 . Hence ϕ represents an infinite order element in $\text{Out}(\pi_1(M))$. \square

The automorphism ϕ of Lemma 15 is analogous to, on the level of fundamental groups, a Dehn twist in surface topology. For each loop c in the center of C , we define a *twist* around a loop c to be an automorphism constructed as above. Up to conjugation, it is the identity on one vertex subgroup and conjugation by an element in the center of the edge subgroup on the other vertex subgroup. By Lemma 15, twists induce infinite order elements of $\text{Out}(\pi_1(M))$.

9.1.2. *The pieces M_i 's have \mathbb{Q} -rank 2.* Although we have been focusing on the case where the M_i 's are irreducible, for this example we choose M_i to be a product of two \mathbb{Q} -rank 1 locally symmetric manifolds, since in this case it is easier to see the main features of the proof for the general case.

Let M be the manifold similar to the example given in the introduction. That is, M is obtained by using D_3 to glue six copies of $S_1 \times S_2$. We take S_j , for $j = 1, 2$, to be the usual compactification of a finite volume, hyperbolic manifolds Y_j (rather than just surfaces) with one cusp whose cross section that is a torus.

Like in the \mathbb{Q} -rank 1 case, M has *twists*, which comes from the center of the fundamental group of each codimension 1 stratum of M . The following is an example.

For each M_i , for $i = 1, 2, \dots, 6$, we denote by $\partial_j M_i$ the boundary stratum the corresponds to $S_j \times \partial S_{j+1}$ (where addition is mod 2). Note that the center of $\pi_1(S_j \times \partial S_{j+1}) \cong \pi_1(S_j) \times \pi_1(\partial S_{j+1})$ is $\pi_1(\partial S_{j+1})$.

Without loss of generality, suppose that $\partial_2 M_1$ is glued to $\partial_2 M_2$. Let c be an element in the center of the fundamental group of $\partial_1 M_1$. Let ϕ be the homomorphism $\pi_1(M) \rightarrow \pi_1(M)$

defined as follows.

$$\phi(g) = \begin{cases} g & \text{if } g \in \pi_1(M_i) \text{ for } i = 3, 4, 5, 6, \\ cgc^{-1} & \text{if } g \in \pi_1(M_i) \text{ for } i = 1, 2. \end{cases}$$

We see that ϕ is a homomorphism since it is a homomorphism on each piece and agrees on the intersections of the pieces.

Lemma 16. *Let M and ϕ be as above. Then ϕ has infinite order in $\text{Out}(\pi_1(M))$.*

Proof. The proof is exactly the same as that of Lemma 15. \square

The main feature of the construction of the above twist is the following. Firstly we pick a codimension 1 stratum $\partial_1 M_1$ of a piece M_1 with center containing a nontrivial element c . Then we define ϕ to be by conjugation by c when restricted to $\pi_1(M_1)$. So the restriction of ϕ to $\pi_1(\partial_1 M_1)$ is the identity, but ϕ restricted to $\pi_1(\partial_j M_1)$ for $j \neq 1$ is not. In this case $j = 2$ only. So we need to define ϕ on pieces adjacent to M_1 along $\partial_j M_1$ (for $j \neq 1$) to be conjugation by c . Then we define ϕ on the pieces adjacent to these accordingly and so on.

9.2. The structure of $\text{Out}(\pi_1(M))$. We now give the set-up in which Theorem 4 is proved.

Let \mathcal{M} be the disjoint union of all the complete locally symmetric spaces corresponding to the locally symmetric pieces M_i in the decomposition of M . By Theorem 14, an element of $\text{Out}(\pi_1(M))$ has to restrict to an isomorphism between vertex groups up to conjugation, which, by the Mostow Rigidity Theorem, is induced by an isometry with respect to the complete, locally symmetric metric on each of the pieces. The restriction map is a homomorphism from $\text{Out}(\pi_1(M))$ to $\text{Isom}(\mathcal{M})$. We call this induced map

$$\eta: \text{Out}(\pi_1(M)) \longrightarrow \text{Isom}(\mathcal{M}).$$

Let $\mathcal{A}(M)$ be the image of $\text{Out}(\pi_1(M))$ under η . We call the elements of $\mathcal{A}(M)$ *turns*. Then $\mathcal{A}(M)$ is the subgroup of $\text{Isom}(\mathcal{M})$ of isometries of \mathcal{M} whose restriction to the boundaries of each pair of strata that are identified in the decomposition of M are homotopic with respect to the gluing.

An isometry of \mathcal{M} permutes the pieces \mathcal{M}_i 's in the decomposition of \mathcal{M} . Thus, for the case where M is of finite type and the decomposition of M has k pieces, there is a homomorphism φ from $\text{Isom}(\mathcal{M})$ to the group of permutations of k letters. Let P be the image of φ . The kernel of φ contains precisely those isometries of \mathcal{M} that preserve each of the components of \mathcal{M} . Hence, $\text{Isom}(\mathcal{M})$ has the structure of an extension of groups as follows.

$$1 \longrightarrow \bigoplus_{i=1}^k \text{Isom}(\mathcal{M}_i) \longrightarrow \text{Isom}(\mathcal{M}) \longrightarrow P \longrightarrow 1.$$

It follows that $\text{Isom}(\mathcal{M})$ is finite since it has a finite index subgroup that is isomorphic to the direct sum of the isometry groups of the components of \mathcal{M} , which are finite. Thus, $\mathcal{A}(M)$ is finite if M is of finite type.

We define the *group of twists* of M to be $\mathcal{T}(M) = \text{Ker } \eta$. Then,

$$1 \longrightarrow \mathcal{T}(M) \longrightarrow \text{Out}(\pi_1(M)) \longrightarrow \mathcal{A}(M) \longrightarrow 1.$$

The group $\mathcal{T}(M)$ is generated by elements called *twists*, which are automorphisms of $\pi_1(M)$ of the following form. The fundamental group $\pi_1(M)$ has the structure of a complex of groups

$G(X)$. Suppose that an edge group G_e has nontrivial center. Let $c \in Z(G_e)$ be a non-identity element. Let v be an end point of e . A piece u is \hat{e} -adjacent to v if u can be connected to v_1 by a finite sequence of chamber s_1, s_2, \dots, s_k such that $u = s_1$, $v = s_k$, and for each i , the chamber s_i and s_{i+1} and the codimension 1 they share is adjacent to the edge NOT of type e . The (e, v) -block is the set of pieces that is \hat{e} -adjacent to v_1 . Let $\{u_j\}$ be the set of vertices the correspond to the (e, v) -block.

Let ϕ be an automorphism of $\pi_1(M)$ that is obtained by gluing the identity automorphism of each vertex groups except for G_{u_j} 's, on which the automorphism to glue is conjugation by c . For the same reasons as in the proof of Lemma 15, the map ϕ is an automorphism that is either trivial or of infinite order. If the (e, v) block is a proper subset of the set of pieces M_i , then ϕ is nontrivial and thus, has infinite order. We call such an automorphism a *twist around the loop c* . At this point, we have only defined twists on the level of fundamental groups. We will see later that twists can be realized as homeomorphisms.

For each edge e such that c belong to the center $Z(G_e)$, if v is an end point of e , then there is a twist along c in the (e, v) -block. Since each twist happens in the union of the chamber in a block, we call the union of all the blocks with nontrivial twists a *twist region*.

Now we prove Theorem 4.

Proof of Theorem 4. We need to show that $\mathcal{T}(M)$ is abelian. To show this, we show that $\mathcal{T}(M)$ is generated by twists and that twists commute.

By definition, the action of a twist on the fundamental group of each of the pieces in the decomposition of M is the identity map up to conjugation by an element in the center of a cusp subgroup. By Mostow rigidity, the image of a twist under η is the identity isometry. Thus, twists belong to $\mathcal{T}(M) = \text{Ker } \eta$.

Now if $\phi \in \text{Ker } \eta$, then the isometry that ϕ induces on \mathcal{M} is the identity map. Hence, for each i , the restriction of ϕ on $\pi_1(M_i)$ is the identity map in $\text{Out}(\pi_1(M_i))$. Let ϕ_{M_i} be the restriction of ϕ to $\pi_1(M_i)$. If M_i and M_j are adjacent pieces that are glued together along S , then the restriction of ϕ_{M_i} to $\pi_1(S)$ is equal to that of ϕ_{M_j} . This implies that ϕ_{M_i} and ϕ_{M_j} differ by a conjugation by an element c in the center of $\pi_1(S)$.

Without lost of generality, assume that ϕ_{M_a} is the identity automorphism of $\pi_1(M_a)$, for some a . Suppose that M_b is adjacent to M_a along S . Let τ_{c_1} be such that ϕ_{M_b} is conjugation by c_1^{-1} . Let τ_{c_1} be the (S, M_b) -twist along c_1 . Let $\phi_1 = \tau_{c_1} \circ \phi$. Then $\phi_1 \in \mathcal{T}(M)$ and the restriction of which to the fundamental groups of M_a and M_b is the identity automorphism. One just needs to see that M_a does not belong to the (S, M_b) -block, which is true because otherwise, ϕ_{M_a} and ϕ_{M_b} cannot differ by a conjugation by elements in $\pi_1(S)$ in the first place.

Now, we repeat this process for another piece that is adjacent to one of the pieces to which the restriction of ϕ_1 is the identity automorphism to get ϕ_2 and so on. It is not hard to see that for each i , the automorphism $\phi_i \in \mathcal{T}(M)$. Also, the twist region of τ_{c_i} does not contain any pieces to which the restriction of ϕ_{i-1} is the identity automorphism. This process terminates at some point if M has finitely many pieces. Therefore, ϕ is a product of twists.

Since twists are defined for each element c in the center of an edge group G_e , it follows that such an element c is also in the center of the highest dimensional cell group that is contained G_e . It is not hard to see that any two twists commute since either they are twists around loops in disjoint twist regions or the loops they twist around commute since they are both in the center of the same highest dimensional cell group.

Hence, $\mathcal{T}(M)$ is a torsion-free abelian group. If M has finitely many pieces, then $\mathcal{T}(M)$ is finitely generated. \square

Topologically, Theorem 4 says that an element $\text{Out}(\pi_1(M))$ is a composition of twists and turns (on the level of fundamental groups). In the next section, we will prove that one can realize these by actual homeomorphisms of M .

10. RIGIDITY OF PIECEWISE LOCALLY SYMMETRIC SPACES

Before proving Theorem 2, we state and prove the properties of the boundary strata of the Borel-Serre compactification of noncompact arithmetic manifolds that we will need.

- a) The cross section of each stratum of an arithmetic manifold M is finitely covered by a manifold \widehat{C} that has the structure of a fiber bundle with fiber a compact nilmanifold F and base a locally symmetric manifold B with nonpositive curvature that can contain a local Euclidean factor. The fundamental group of \widehat{C} has the structure given by the following extension

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(\widehat{C}) \longrightarrow \pi_1(B) \longrightarrow 1,$$

where $\pi_1(F) = \Gamma_{N_P}$, and $\pi_1(B)$ is the fundamental group of a compact, nonpositively curved, locally symmetric space. By the Godement compactness criterion (Theorem 5.30 in [17]), the group $\pi_1(F)$ is nontrivial.

- b) The center of $M_P N_P$ is contained in N_P .

Let P be a parabolic subgroup of G as in Section 2. Let $L_P = P/N_P$ be the Levi quotient of P . Then both N_P and L_P are rational algebraic groups. The group $L_P \cong A_P M_P$. Let

$$\Gamma_P = \Gamma \cap P \quad \text{and} \quad \Gamma_{N_P} = \Gamma \cap N_P.$$

Then Γ_P is an arithmetic subgroup of N_P and N_P/Γ_{N_P} is compact ([17]). The projection of Γ_P in L_P is an arithmetic group of M_P ([3]) denoted by Γ_{M_P} . Then Γ has the structure given by the short exact sequence

$$1 \longrightarrow \Gamma_{N_P} \longrightarrow \Gamma_P \longrightarrow \Gamma_{M_P} \longrightarrow 1$$

that need not split. The group Γ_{N_P} is the fundamental group of a compact nilmanifold and Γ_{M_P} is the fundamental group of a compact locally symmetric space with nonpositive curvature.

Proof of property (a). This follows from the Langlands decomposition of \mathbb{Q} parabolic subgroups of G . To see this, let Λ be a finite index, torsion-free subgroup of Γ_{M_P} . Then $\Gamma_{N_P} \rtimes \Lambda$ corresponds to a finite-sheeted cover \widehat{C} of C that has the structure of a fiber bundle with fiber $F = N_P/\Gamma_{N_P}$ and base $B = X_P/\Lambda$. The group Γ_{N_P} is a cocompact lattice in N_P (see [17]). The manifold X_P/Λ is a compact, locally symmetric, nonpositively curved manifold since Λ is torsion-free. \square

Proof of Property (b). Firstly, we show that any element g in the identity component Z^0 of the center Z of $M_P N_P$ that is not in N_P must commute with the Borel subgroup of G . This will be a contradiction since the centralizers of Borel subgroups are trivial. Then we will show that Z must be connected.

Let T be a maximal \mathbb{R} -split torus and let \mathfrak{a} be the Lie algebra of T . Then the Lie algebra of the unipotent radical N_P of P is

$$\mathfrak{n}_I = \sum_{\alpha \in \Phi^+ \setminus \Phi_I} g_\alpha,$$

where Φ^+ is the set of positive roots and $\Phi_I \subset \Phi^+$ is a set of simple roots. Let

$$\mathfrak{a}_I = \bigcap_{\alpha \in I} \ker \alpha.$$

Let \mathfrak{a}^I be the orthogonal complement of \mathfrak{a}_I . Then the Lie algebra of M_P is

$$\mathfrak{m}_I = \mathfrak{a}^I \oplus \sum_{\alpha \in \Phi_I} g_\alpha \oplus l,$$

for l a subspace of the centralizer of \mathfrak{a} . (See [3] for details of the above). Now, since T normalizes $M_P N_P$, it normalizes Z . Hence, the Lie algebra \mathfrak{z} of Z^0 is a direct sum of subspaces of root spaces g_α . If \mathfrak{z} contains elements x in a root space g_β not in Φ_I . Observe that β cannot be 0, for g_β commutes with g_α for all $\alpha \in \Phi_I$, which means that $g_\beta \subset \mathfrak{a}_I$, which intersects with \mathfrak{m}_I trivially.

If $\beta \in \Phi_I$, then x commutes with $\mathfrak{n} \oplus \mathfrak{m}$ and \mathfrak{a}_I since $\mathfrak{a}_I \subset \ker g_\beta$. Therefore, x commutes with $\Phi^+ \oplus \mathfrak{a}^I$. Thus x commutes with a Borel subgroup contained in P . Since the centralizer of any Borel subgroup is trivial, this is a contradiction. So \mathfrak{z} is contained in \mathfrak{n}_I . Hence Z^0 is contained in the unipotent radical N_P of P .

If $g_\beta \subset l$, then x commutes with \mathfrak{a} (since l is a subspace of the centralizer of \mathfrak{a}_I) and Φ^+ . Hence x commutes with the Borel subgroup and again, we get a contradiction.

Now, we prove that Z is connected. Suppose it is not. Then there is an element g in Z such that $g = mn$, for $n \in N_P$ and $m \in M_P - \{1\}$. Let X_P be the symmetric space $(K \cap M_P) \backslash M_P$. Then m acts on X_P , and the action of m commutes with $M_P(\mathbb{Z})$, which is a lattice of M_P . Thus m commutes with the isometry group of X_P .

Since X_P is isometrically a product of symmetric space of noncompact type Y and a Euclidean space E , it follows that m must split into a product of two isometries, each of which preserves either Y or E . Hence, m acts by a translation on E and by the identity on Y . Since each translation can be connected to the identity via a one parameter subgroup (namely translation with the same direction but different displacement), this means that if the above translation is nontrivial, then there is vector in \mathfrak{z} that is not in \mathfrak{n}_I , which is a contradiction to above. Hence, m acts on X_P trivially, which implies that $\exp(m) \in K \cap M_P$. This means that m is an isometry of X_P the preserves the fiber N_P . Thus, the restriction of m to each fiber has to be an isometry.

By [14, Theorem 9], any isometry of a simply connected, 2-step nilpotent Lie group that commute with a lattice has to be left multiplication by an element of the center. The proof of the theorem can be extended easily to the general case of a simply connected nilpotent Lie group. Hence, the element m has to be in the center of N_P , which is a contradiction to the assumption that $m \in M_P$. Therefore, Z^0 contains elements outside N_P , which is a contradiction. \square

Now we prove Theorem 2.

Proof of Theorem 2. Let $\phi: \pi_1(M) \longrightarrow \pi_1(M)$ be an isomorphism. We are going to construct a homeomorphism from M to M by firstly defining homeomorphisms on each piece in the decomposition of M and then gluing them together in such a way that the resulting homeomorphism induces the isomorphism ϕ .

Let M_i , for i in some index set I , be the pieces in the decomposition of M . By Lemma 14, the map ϕ defines a bijection $\alpha: I \longrightarrow I$ such that $\phi(\pi_1(M_i)) = \pi_1(M_{\alpha(i)})$ up to conjugation. By the Mostow Rigidity Theorem, the restriction of ϕ on each vertex group of $\pi_1(M)$ up to conjugation is induced by an isometry f_i from M_i to $M_{\alpha(i)}$. For each i , the isometry f_i extends to a homeomorphism between manifolds with corners from M_i to $M_{\alpha(i)}$ ([10, Proposition III.5.13]), which we will also call f_i .

At this point one may want to glue the isometries f_i 's together and claim that it is the desired homeomorphism of M . However, there are two problems. One is that the gluing of f_i at each pair of boundaries of M_i 's that are glued together might not be compatible with the gluing. The other problem is that if M_i and M_j are adjacent pieces (that is, they share a codimension 1 stratum), it may happen that f_{i*} and f_{j*} define different isomorphisms (by a conjugate) on the fundamental group of M_i and M_j even though they agree on the common boundary stratum. See Lemma 15 above for such an example of ϕ .

Hence we need to do some modifications to f_i 's near the boundary of M_i before gluing them together. Specifically, it suffices to show the following two things. Firstly, if M_i and M_j are adjacent pieces whose intersection is S , then there is an isotopy from f_i (restricted to S) to f_j (restricted to S). Once we have this we can modify the maps f_i and f_j by an isotopy of S on a tubular neighborhood of S that is compatible with the gluing of M . However, the induced map of the obtained homeomorphism may be different from ϕ by a twist. Thus secondly, we need to realize twists as homeomorphisms by realizing the change of basepoint given by the conjugation difference between f_{i*} and the restriction of ϕ to the given stratum. It suffices to show that on a twist along a loop c in the center of $\pi_1(S)$ for some codimension 1 stratum S , there is an isotopy of S that moves the basepoint around a loop homotopic to c . So the two problems boil down to proving the same thing, that is, the following theorem, which we will assume for now and give the proof after this.

Theorem 17. *Let M be a piecewise locally symmetric manifold that has one stratum S . If $f: \tilde{S} \longrightarrow \tilde{S}$ is a $\pi_1(S)$ -equivariant isometry that is $\pi_1(S)$ -equivariantly homotopic to the identity map of \tilde{S} , then f is $\pi_1(S)$ -equivariantly isotopic to the identity map. Thus, if $g \in Z(\pi_1(S))$, then there is an isotopy on S that moves the base point around a loop that is homotopic to g .*

Assuming Theorem 17, we observe that the restriction of f_i and f_j to each pair of boundary components of M_i and M_j that are identified, say S , induces the same isomorphism of $\pi_1(S)$ up to conjugation. Since S is aspherical, the restriction of f_i and f_j to S are homotopic isometries. By Theorem 17, we can modify the maps f_i and f_j by an isotopy of S on a tubular neighborhood of the corresponding boundary component of M_i and M_j that is compatible with the gluing of M and the action of ϕ on fundamental groups. Let $f: M \longrightarrow M$ be the homeomorphisms obtained by gluing the modified f_i . Then $f_* = \phi$.

Now we construct the homeomorphism

$$\rho: \text{Out}(\pi_1(M)) \longrightarrow \text{Homeo}(M)/\text{Homeo}_0(M).$$

To do this, we need to slightly modify the decomposition of M into pieces. Instead of taking M as a union of compact manifolds with corners M_i glued along pairs of diffeomorphic strata, for each codimension k stratum P of an M_i , we glue $P \times [0, 1]^k$ to P along $P \times (0, 0, \dots, 0)$. If $Q \subseteq P$ is codimension $k + 1$ stratum, then we glue $Q \times [0, 1] \times (0, 0, \dots, 0)$ to $P \times (0, 0, \dots, 0)$. The space N_i we obtain is a compact manifold with corners that is diffeomorphic to M_i . Hence M is a union of N_i glued to each other in the same way as the M_i 's are. We will take this decomposition of M into N_i 's for the rest of the proof.

For each $\varphi \in \text{Out}(\pi_1(M))$, we define $\rho(\varphi)$ to be the homeomorphism that takes M_i isometrically onto $M_{\varphi(i)}$. Any twist and turn will happen on the tubes connecting the boundary strata as follows. If S is a codimension k stratum, one takes the product of the corresponding k straight line isotopies as in the proof of Theorem 17 that realizes half of the amount that needs to be twisted.

We claim that ρ is a homomorphism. To see this, for each $\varphi \in \text{Out}(\pi_1(M))$, the homeomorphism $\rho(\varphi)$ is the unique homeomorphism that induces φ that is an isometry when restricted to each M_i , and that is the straight line isotopy on each tube. The composition of any two such maps is a map of the same type. By uniqueness of such maps, ρ is a homomorphism. \square

Proof of Theorem 17. We have, as before

$$1 \longrightarrow F \longrightarrow C \longrightarrow B \longrightarrow 1,$$

where F is the torsion-free, discrete, cocompact subgroup of the isometry group of a simply connected Riemannian nilpotent Lie group H (i.e. H is endowed with a left-invariant Riemannian metric), which is the unipotent radical of the parabolic P corresponding to C .

If f is an isometry of S that is homotopic to the identity map, then if \tilde{f} is a lift of f to the universal cover \tilde{S} of S , then \tilde{f} commutes with all the deck transformation of \tilde{S} . By the proof of Property (b) above, the map \tilde{f} is in the center of the unipotent radical N_P of P .

Since, N_P is a nilpotent, simply connected Lie group, the exponential map is a global diffeomorphism from \mathfrak{n}_P to N_P . In logarithmic coordinates of N_P , one can take the straight line in N_P from the identity to \tilde{f} . Since \tilde{f}_t is a straight line path in the Lie algebra of the nilpotent Lie group N_P , and N_P is normal in P , it follows that conjugation by $\gamma \in C$ preserves this path. Since $\tilde{f}_1 = \tilde{f}$ commutes with C , it follows that the same is true for \tilde{f}_t for a given $t \in [0, 1]$. That is,

$$\tilde{f}_t \circ \gamma = \gamma \circ \tilde{f}_t, \quad \text{for all } \gamma \in C.$$

So multiplication by \tilde{f}_t in $M_P N_P$ is an C -equivariant isotopy. Hence, it descends to an isotopy on S . \square

11. APPENDIX: COMPLEXES OF GROUPS

The theory of complexes of groups was developed as a natural generalization of the Bass-Serre theory of graphs of groups. There are, in general, two perspectives one can take in Bass-Serre theory. One is that in the spirit of van Kampen's theorem. The other is to view it as an inverse problem of groups acting on trees. The theory of complexes of groups seems to have been developed with the latter point of view.

For the manifolds in this paper, i.e., piecewise locally symmetric spaces, the first point of view is more natural and geometric. In this section, we are not going to develop the whole

theory of complexes of groups from scratch, but we are going to present the first perspective for the complexes of groups as a guiding principle of geometric understanding. We are going to review the theory of complexes of groups in the context of this paper. Although this may not be of the greatest generality as the theory in the literature, it gives all that is needed in this paper, and the author believes that this captures the heart of complexes of groups by looking at concrete, simpler settings.

A *convex polytope* P in an affine space \mathcal{A} is the convex hull of a finite subset. The *dimension* of P is the dimension of the affine subspace that it spans.

A *convex cell complex* Y is a collection Λ of convex polytopes in an affine space \mathcal{A} such that

- i) if $P \in \Lambda$ and F is a face of P , then $F \in \Lambda$ and
- ii) for any two polytopes P and Q in Λ , either $P \cap Q = \emptyset$ or $P \cap Q$ is a common face of both polytopes.

The barycentric subdivision of a convex cell complex Y is an ordered simplicial complex bY . An *ordered simplicial complex* X is a simplicial complex in which there is a partial order on the set of vertices $V(X)$ such that X is the geometric realization of the simplicial complex $\text{Flag}(V(X))$, the set of finite chains of $V(X)$ (see [6, Appendix A] for the geometric realization of the simplicial complex $\text{Flag}(P)$ for a partially ordered set P). The partial order on $V(X)$ gives each edge in X a direction. Two edges a and b are composable if $t(b) = i(a)$, where t denotes the terminal point and i denotes the initial point of an edge.

Let Y be a convex cell complex. A *complex of spaces* over Y is a connected CW complex K and a map $p: K \rightarrow Y$ such that

- i) for each open cell e_α of Y , the pre-image $p^{-1}(e_\alpha)$ is a connected subcomplex of K of the form $K_{e_\alpha} \times e_\alpha$, where K_{e_α} is the pre-image of the center of the cell e_α ,
- ii) if e_β is a cell contained in the boundary of e_α , the induced map on fundamental groups $\pi_1(K_{e_\beta}) \rightarrow \pi_1(K_{e_\alpha})$ is injective.

If K is a complex of spaces over Y , then there is an obvious bijection between the set of vertices of bY and the set of spaces K_{e_α} , for some cell e_α in Y . By the van Kampen theorem, the fundamental group $\pi_1(K)$ depends only on the fundamental groups of the spaces K_{e_α} , the complex Y , and the maps $\pi_1(K_{e_\beta}) \rightarrow \pi_1(K_{e_\alpha})$ whenever they are defined. This motivates the definition of complexes of groups and their fundamental groups.

A *complex of groups* $G(X)$ on an ordered simplicial complex X is given by the following data.

- 1) For each $\sigma \in V(X)$, a group G_σ , called a *local group*, is associated to σ .
- 2) For each $a \in E(X)$, an injective homomorphism $\psi: G_{i(a)} \rightarrow G_{t(a)}$.
- 3) For each pair $a, b \in E(X)$ of composable edges, an element $g_{a,b} \in G_{t(a)}$ such that
 - a) $\text{Ad}(g_{a,b})\psi_{ab} = \psi_a\psi_b$.
 - b) $\psi_ag_{b,c}g_{a,bc} = g_{a,b}g_{a,bc}$ for each triple of composable edges $a, b, c \in E(X)$. This is called the *cocycle condition*.

If K is complex of spaces over a complex Y , then associated to K there is a complex of groups over $X := bY$. It is geometrically clear where the first two conditions in the definition of complexes of groups come from. Condition (3a) corresponds to how the fiber corresponding to a 2-dimensional cell is included into the graph of spaces corresponding to the 1-skeleton of X . One can interpret $g_{a,b}$ as a change of base point, that is, the base point difference between two different ways of including $K_{i(b)}$ into $K_{t(a)}$: one is via $\psi_a\psi_b$, and the other is directly via

ψ_{ab} . Condition (3b), or the cocycle condition, is the consistency condition that correspond to the change of basepoint between different inclusions of a higher dimensional cell group to a lower dimension cell group.

The *geometric realization* of a complex of groups $G(X)$ is a complex of spaces $K(X)$ over a complex Y whose barycentric subdivision bY is X such that the associated complex of groups is $G(X)$. The *fundamental group of a complex of groups* $G(X)$, denoted by $\pi_1(G(X))$ is the fundamental group of its geometric realization. The *aspherical realization* or *classifying space* $BG(X)$ of a complex of groups $G(X)$ is a geometric realization of $G(X)$ such that the fiber over each point is aspherical. For a given complex of group $G(X)$, the classifying space $BG(X)$ always exists and the homotopy type of its classifying spaces do not depend on the choice of classifying spaces [9].

The nerve of a piecewise locally symmetric space M is a convex cell complex Y . Let X be the barycentric subdivision bY . The manifold M has the structure of a complex of spaces since the inclusion of one stratum into another is injective on fundamental groups. Hence, $\pi_1(M)$ is the fundamental group of the associated complex of groups, say, $G(X)$. Then the vertex groups of $G(X)$ correspond to the fundamental groups of the strata of M . Since the fiber over each point of Y in M is an aspherical manifold, M is an aspherical realization of the complex of groups $G(X)$, that is, M is a $BG(X)$.

The fundamental group of a complex of groups $G(X)$ has a presentation as follows. Let T be a maximal tree in the 1-skeleton of X . The generators are those in each local group G_σ , for $\sigma \in V(X)$, and all edges $a \in E(X)$, the set of edges of X . The relations are

- a) the relations in the group G_σ 's,
- b) for each $a \in E(X)$ and $h \in G_{i(a)}$, then $\psi_a(h) = a^{-1}ha$,
- c) for each pair of composable edges a and b with $c = ab$, then $c = bag_{a,b}$,
- d) $a = 1$ for $a \in T$

Let $G(X)$ be a complex of groups over X . Like in Bass-Serre theory, where the universal cover of a graph of spaces is a graph of spaces over a simply connected graph (i.e. a tree, called the Bass-Serre tree associated to the given graph of groups), the universal cover $\widetilde{BG(X)}$ of $BG(X)$ is a simply connected complex of spaces over a simply connected complex \widetilde{X} of the same dimension as X . In the same way as the fundamental group of a graph of groups acts without inversions on the Bass-Serre tree with quotient the original graph, the group $\pi_1(G(X))$ acts without inversions on \widetilde{X} with quotient X .

However, unlike in the theory of graphs of groups, the inclusion of G_σ into the global group $\pi_1(G(X))$ needs not be injective. The stabilizer of a cell $\tilde{\sigma}$ in \widetilde{X} that projects to a cell σ is not, in general, isomorphic to G_σ , but is isomorphic to the image of G_σ under the canonical inclusion of G_σ into $\pi_1(G(X))$. This gives rise to the notion of *developability* for a complex of groups.

A complex of groups is called *developable* if the inclusion of each vertex group G_σ into the global group $\pi_1(G(X))$ is injective. This is equivalent to the usual notion of developable, as in [9], which is that there exists a complex Y such that X is the quotient of Y under the action of a group without inversion. It follows that for a developable $G(X)$, the stabilizer in $\pi_1(G(X))$ of a cell $\tilde{\sigma}$ in \widetilde{X} that projects to a cell σ is a conjugate of G_σ , which leads to the notion of local development of a vertex $\sigma \in V(X)$.

For each $\sigma \in X$, the *local development* of σ is the following complex $\text{st}(\tilde{\sigma})$. Firstly we define a complex $\text{St}(\tilde{\sigma})$. The set of vertices of $\text{St}(\tilde{\sigma})$ is one-to-one correspondence with set D_σ of cosets of G_δ , for some subgroup G_δ of G_σ and G_λ that contains G_σ . Give D_σ a partial ordering by inclusion. The complex $\text{St}(\tilde{\sigma})$ is the geometric realization of $\text{Flag}(D_\sigma)$. Let $\tilde{\sigma}$ be the vertex corresponding to G_σ . We define $\text{st}(\tilde{\sigma})$ to be the star of $\tilde{\sigma}$ in $\text{St}(\tilde{\sigma})$, which is defined to be the union of the interior of the simplices in X that meet $\tilde{\sigma}$. The group G_σ acts on $\text{st}(\tilde{\sigma})$ in the obvious way with quotient the star of σ in X .

If X is given a geodesic metric d , then this metric lifts to a geodesic metric \tilde{d} on \tilde{X} . For example, for the case of a piecewise locally symmetric space M , the complex X naturally admits a piecewise Euclidean metric. In general, suppose that X is an M_κ -polyhedral complex, that is, each simplex in X is metrized as a simplex in M_κ , the simply connected, complete, Riemannian manifold with constant curvature κ . The metric on each simplex of X induces a metric on $\text{st}(\tilde{\sigma})$ for each $\sigma \in X$ that makes $\text{st}(\tilde{\sigma})$ an M_κ -polyhedral complex. If for each $\sigma \in V(X)$, the local development $\text{st}(\tilde{\sigma})$ has curvature $\leq \kappa$, then we say that $G(X)$ has curvature $\leq \kappa$.

Theorem 18 ([4]). *If a complex of group $G(X)$ is nonpositively curved, that is, it has curvature ≤ 0 , then it is developable.*

If $G(X)$ is developable, then $BG(X)$ is homotopy equivalent to \tilde{X} . To see this, consider the universal cover $\widetilde{BG(X)}$ of $BG(X)$. If the inclusions of local groups are injective, then for each $\sigma \in V(X)$, each connected component of pre-image of K_σ in $\widetilde{BG(X)}$ is a copy of the universal cover \tilde{K}_σ , which is contractible. So $\widetilde{BG(X)}$ is a complex of contractible spaces over \tilde{X} . It follows that $\widetilde{BG(X)}$ is homotopy equivalent to the base complex \tilde{X} (see [6, Lemma E.3.3]). Therefore, $\widetilde{BG(X)}$ is contractible if and only if \tilde{X} is.

A sufficient condition for the complex \tilde{X} to be contractible is that it is CAT(0). For the case of piecewise locally symmetric spaces, the simplices in \tilde{X} are Euclidean convex polyhedron. So the associated complex of groups $G(X)$ is over a M_0 -complex. Suppose that for each $\sigma \in X$, the local development of σ has curvature ≤ 0 . Then $G(X)$ is nonpositively curved, and thus, is developable by the above theorem. This also implies that \tilde{X} has curvature ≤ 0 since being nonpositively curved is a local condition. Moreover, \tilde{X} is simply connected, and thus is CAT(0) if it is nonpositively curved, and thus is contractible. It then follows that $BG(X)$ is contractible.

For a M_0 -complex X , it is nonpositively curved if the link condition is satisfied. An M_κ -polyhedral complex is said to satisfied the *link condition* if for each vertex $\sigma \in X$, the link complex $\text{Lk}(\sigma, X)$ is a CAT(1) space.

Theorem 19 ([4]). *An M_κ -polyhedral complex X with finite shape (i.e. there are finitely many isometry classes of simplices of X) has curvature $\leq \kappa$ if and only if it satisfies the link condition.*

REFERENCES

1. L. Auslander, *Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups*, Ann. of Math. (2) **71** (1960), 579–590.
2. A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.

3. Armand Borel and Lizhen Ji, *Compactifications of locally symmetric spaces*, J. Differential Geom. **73** (2006), no. 2, 263–317.
4. Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.
5. Jon Michael Corson, *Complexes of groups*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 199–224. MR 1162493 (93h:57003)
6. Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.
7. Benson Farb, *Group actions and Helly's theorem*, Adv. Math. **222** (2009), no. 5, 1574–1588.
8. F. T. Farrell and L. E. Jones, *Rigidity and other topological aspects of compact nonpositively curved manifolds*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), no. 1, 59–64.
9. André Haefliger, *Extension of complexes of groups*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 275–311.
10. Lizhen Ji, *Compactifications of symmetric spaces and locally symmetric spaces*, Geometric complex analysis (Hayama, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 297–308.
11. Seonhee Lim and Anne Thomas, *Covering theory for complexes of groups*, J. Pure Appl. Algebra **212** (2008), no. 7, 1632–1663.
12. S. Morita, *Geometry of characteristic classes*, Translations of Mathematical Monographs, vol. 199, American Mathematical Society, Providence, RI, 2001, Translated from the 1999 Japanese original, Iwanami Series in Modern Mathematics.
13. G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. (2) **60** (1954), 1–27.
14. T. T. Nguyen Phan, *Smooth (non)rigidity of cusp-decomposable manifolds (to appear in comment. math. helv.)*.
15. ———, *Smooth (non)rigidity of piecewise rank one manifolds (preprint)*.
16. John R. Stallings, *Non-positively curved triangles of groups*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 491–503.
17. D. Witte Morris, *Introduction to arithmetic groups*.
18. Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

DEPARTMENT OF MATHEMATICS, 5734 S. UNIVERSITY AVE., CHICAGO, IL 60637

E-mail address: ttamnp@math.uchicago.edu